

## AROUND MULTIVARIATE SCHMIDT-SPITZER THEOREM

PER ALEXANDERSSON AND BORIS SHAPIRO

**ABSTRACT.** Given an arbitrary complex-valued infinite matrix  $\mathcal{A} = (a_{ij})$ ,  $i = 1, \dots, \infty; j = 1, \dots, \infty$  and a positive integer  $n$  we introduce a naturally associated polynomial basis  $\mathfrak{B}_{\mathcal{A}}$  of  $\mathbb{C}[x_0, \dots, x_n]$ . We discuss some properties of the locus of common zeros of all polynomials in  $\mathfrak{B}_{\mathcal{A}}$  having a given degree  $m$ ; the latter locus can be interpreted as the spectrum of the  $m \times (m+n)$ -submatrix of  $\mathcal{A}$  formed by its  $m$  first rows and  $(m+n)$  first columns. We initiate the study of the asymptotics of these spectra when  $m \rightarrow \infty$  in the case when  $\mathcal{A}$  is a banded Toeplitz matrix. In particular, we present and partially prove a conjectural multivariate analog of the well-known Schmidt-Spitzer theorem which describes the spectral asymptotics for the sequence of principal minors of an arbitrary banded Toeplitz matrix. Finally, we discuss relations between polynomial bases  $\mathfrak{B}_{\mathcal{A}}$  and multivariate orthogonal polynomials.

## 1. INTRODUCTION

The approach of this paper is motivated by the modern interpretation of the Heine-Stieltjes multiparameter spectral problem as presented in [9] and [10]. Let us recall some relevant results in the matrix set-up.

Given integers  $m > 0$  and  $n \geq 0$  consider the space  $Mat(m, m+n)$  of complex-valued  $m \times (m+n)$ -matrices. For  $s = 0, \dots, n$  define the  $s$ -th unit matrix

$$\mathcal{I}_s := (\delta_{s+i-j}) \in Mat(m, m+n).$$

(In what follows the sizes of matrices can be infinite.)

**Definition 1** (see [10]). *Given a matrix  $A \in Mat(m, m+n)$  define its eigenvalue locus  $\mathcal{E}_A$  as*

$$\mathcal{E}_A := \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1} : \text{rank} \left( A - \sum_{s=0}^n x_s \mathcal{I}_s \right) < m \right\}.$$

For  $n = 0$ ,  $\mathcal{E}_A$  coincides with the usual set of eigenvalues of a square matrix  $A$ .

**Proposition 2** (Lemma 1 of [10]). *For arbitrary  $A \in Mat(m, m+n)$  the eigenvalue locus  $\mathcal{E}_A$  consists of  $\binom{m+n}{n+1}$  points counting multiplicities. In other words, counting multiplicities there exist  $\binom{m+n}{n+1}$  eigenvalue tuples  $(x_0, x_1, \dots, x_n)$  such that  $A - \sum_{s=0}^n x_s \mathcal{I}_s$  has rank smaller than  $m$ .*

**Remark 3.** *Notice that for  $n > 0$ , the locus  $\mathcal{E}_A$  is not a complete intersection since it is given by the vanishing of all maximal minors of  $A$ . (A similar phenomenon can be observed for common zeros of multivariate Schur polynomials, since Schur polynomials are given by determinant formulas.)*

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**Notation 4.** Given an infinite matrix  $\mathcal{A} = (a_{ij})$ ,  $i = 1, \dots, \infty; j = 1, \dots, \infty$ , an integer  $n \geq 0$ , and an  $m$ -tuple of positive integers  $I = (i_1, i_2, \dots, i_m)$  satisfying  $1 \leq i_1 < i_2 < \dots < i_m \leq m + n$ , consider the submatrix  $A_I$  of  $\mathcal{A} - \sum_{s=0}^n x_s \mathcal{I}_s$  formed by the first  $m$  rows and  $m$  columns indexed by  $I$ . Define

$$(1) \quad P_{\mathcal{A}}^I(x_0, x_1, \dots, x_n) := \det A_I.$$

Evidently,  $P_{\mathcal{A}}^I(x_0, \dots, x_n)$  is a maximal minor of the principal  $m \times (m + n)$  submatrix of  $\mathcal{A} - \sum_{s=0}^n x_s \mathcal{I}_s$  formed by its  $m$  first rows and  $m + n$  first columns. Therefore  $P_{\mathcal{A}}^I(x_0, \dots, x_n)$  is a polynomial in  $x_0, \dots, x_n$ .

**Proposition 5.** In the above notation the following holds:

- (i) for any multiindex  $I$  with  $|I| = m$ ,  $\deg P_{\mathcal{A}}^I(x_0, \dots, x_n) = m$ ;
- (ii) all  $\binom{m+n}{m}$  polynomials  $P_{\mathcal{A}}^I(x_0, \dots, x_n) \in \mathbb{C}[x_0, \dots, x_n]$  with  $|I| = m$  are linearly independent which implies that the totality of all polynomials  $P_{\mathcal{A}}^I(x_0, \dots, x_n)$  is a linear basis of  $\mathbb{C}[x_0, \dots, x_n]$ ;
- (iii) the set  $\mathcal{E}_{\mathcal{A}}^{(m)}$  of common zeros of all  $P_{\mathcal{A}}^I(x_0, \dots, x_n)$  with  $|I| = m$  is a finite subset of  $\mathbb{C}^{n+1}$  of cardinality  $\binom{m+n}{n+1}$  counting multiplicities.

**Remark 6.** Notice that for  $\binom{m+n}{m}$  randomly chosen polynomials in  $\mathbb{C}[x_0, x_1, \dots, x_n]$  of degree  $m$  the set of their common zeros is typically empty.

Proposition 5 together with our numerical experiments motivate the following question.

Given an arbitrary infinite matrix  $\mathcal{A}$  as above, associate to each  $\mathcal{E}_{\mathcal{A}}^{(m)}$  its “root-counting” measure  $\mu_{\mathcal{A}}^{(m)}$  supported on  $\mathcal{E}_{\mathcal{A}}^{(m)} \subset \mathbb{C}^{n+1}$  by assigning to every point  $p \in \mathcal{E}_{\mathcal{A}}^{(m)}$  the point mass  $\kappa(p) / \binom{m+n}{n+1}$  where  $\kappa(p)$  is the multiplicity of  $p$ . (Obviously,  $\mu_{\mathcal{A}}^{(m)}$  is a discrete probability measure.)

**Main Problem.** Under which assumptions on  $\mathcal{A}$  does the weak limit  $\mu_{\mathcal{A}} = \lim_{m \rightarrow \infty} \mu_{\mathcal{A}}^{(m)}$  exists? In case when  $\mu_{\mathcal{A}}$  exists, is it possible to describe the support and density of the measure?

In the classical case  $n = 0$ , the above problem was intensively studied by many authors. The main focus has been when  $\mathcal{A}$  is either a Jacobi or a Toeplitz matrix (or their generalizations such as block-Toeplitz matrices etc.), see e.g. [3, 4, 11, 12].

The main goal of this note is to present a multivariate analogue of the well-known theorem by P. Schmid and F. Spitzer [8], where they describe  $\mu_{\mathcal{A}}$  for an arbitrary banded Toeplitz matrix  $\mathcal{A}$  in the case  $n = 0$ .

Namely, let  $\mathcal{A} = (c_{i-j})$ , with  $i, j = 1, 2, \dots$  be an infinite, banded Toeplitz matrix, where  $c_i = 0$  if  $i < -k$  or  $i > h$ . Fixing  $n \geq 0$  as above, we obtain for each positive integer  $m$  the eigenvalue locus  $\mathcal{E}_{\mathcal{A}}^{(m)}$  of the principal  $m \times (m + n)$  submatrix  $A^{(m)}$  of  $\mathcal{A}$ .

Define the *limit set*  $B_{\mathcal{A}}$  of eigenvalue loci as

$$(2) \quad B_{\mathcal{A}} = \left\{ \mathbf{x} \in \mathbb{C}^{n+1} : \mathbf{x} = \lim_{m \rightarrow \infty} \mathbf{x}_m, \mathbf{x}_m \in \mathcal{E}_{\mathcal{A}}^{(m)} \right\}, \quad \mathbf{x} = (x_0, \dots, x_n).$$

In other words,  $B_{\mathcal{A}}$  is the set of limit points of the sequence  $\{\mathcal{E}_{\mathcal{A}}^{(m)}\}$ . Thus  $B_{\mathcal{A}}$  is the support of the limiting measure  $\mu_{\mathcal{A}}$  if it exists. (For a general infinite matrix  $\mathcal{A}$  as above, its limit set  $B_{\mathcal{A}}$  might be empty.)

Set

$$(3) \quad Q(t, \mathbf{x}) = t^k \left( \sum_{j=-k}^h c_j t^j - \sum_{j=0}^n x_j t^j \right),$$

and let  $\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x}), \dots, \alpha_{k+h}(\mathbf{x})$  be the roots of  $Q(t, \mathbf{x}) = 0$ , ordered according to their absolute values, i.e.  $|\alpha_i(\mathbf{x})| \leq |\alpha_{i+1}(\mathbf{x})|$  for all  $0 < i < k + h$ . Let  $C_{\mathcal{A}}$  be the real semi-algebraic set given by the condition:

$$(4) \quad C_{\mathcal{A}} = \{\mathbf{x} \in \mathbb{C}^{n+1} : |\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \dots = |\alpha_{k+n+1}(\mathbf{x})|\}.$$

Our main conjecture is as follows.

**Conjecture 7.** *For any banded Toeplitz matrix  $\mathcal{A}$ , if  $B_{\mathcal{A}}$  is defined by (2) and  $C_{\mathcal{A}}$  defined by (4) then  $B_{\mathcal{A}} = C_{\mathcal{A}}$ .*

By Conjecture 7 the set  $B_{\mathcal{A}}$  is a real semi-algebraic  $(n+1)$ -dimensional subset of  $\mathbb{C}^{n+1}$ . In the classical case  $n = 0$ , Conjecture 7 is settled by P. Schmidt and F. Spitzer in [8]. Another important case when Conjecture 7 has been proved follows from some known results on multivariate Chebyshev polynomials, which is presented in Example 8 below. Namely, when  $k = 1$  and  $h = n + 1$  with  $c_{-1}$  and  $c_{n+1}$  non-zero, we may do a affine change of the variables and a scaling of  $\mathcal{A}$ . This reduces to the latter case to  $c_{-1} = c_{n+1} = 1$  and all other  $c_i = 0$ .

For these particular values, the family  $\{P_{\mathcal{A}}^I(\mathbf{x})\}$  becomes the multivariate Chebyshev polynomials of the second kind, see e.g. [5, 7, 2, 13]. These polynomials also have a close connection to another well-known family of polynomials, namely the Schur polynomials.

**Example 8.** For the above matrices corresponding to the multivariate Chebyshev polynomials the eigenlocus  $\mathcal{E}_{\mathcal{A}}^{(m)}$  can be described explicitly, see for example [6].

More precisely, the points in  $\mathcal{E}_{\mathcal{A}}^{(m)}$  lie on a real,  $n$ -dimensional surface  $C_{\mathcal{A}} \subset \mathbb{C}^{n+1}$  which is naturally parametrized by an  $(n+1)$ -dimensional torus  $T^{n+1}$ . This parametrization is given by

$$(5) \quad C_{\mathcal{A}} = \{\mathbf{x} \in \mathbb{C}^{n+1} | x_j = -e_{j+1}(\exp(i\theta_1), \dots, \exp(i\theta_{n+1}), \exp(i\theta_{n+2}))\}$$

where  $(\theta_1, \dots, \theta_{n+1})$  lie on  $T^{n+1}$ ,  $\sum_{j=0}^{n+2} \theta_j = 0$ , and  $e_j$  is the  $j$ -th elementary symmetric function in  $n+2$  variables.

Notice that for  $\mathbf{x} \in C_{\mathcal{A}}$ ,

$$(6) \quad Q(t, \mathbf{x}) = 1 + x_0 t + x_1 t^2 + \dots + x_n t^{n+1} + t^{n+2} =$$

$$(7) \quad = \prod_j (t + e^{i\theta_j})$$

by the Vieta formulas. Thus, for  $\mathbf{x} \in C_{\mathcal{A}}$ , all roots of  $Q$ , (as a polynomial in  $t$ ) have absolute value equal to 1 when the  $x_j$  are parametrized as in (5).

Furthermore, the points in  $\mathcal{E}_{\mathcal{A}}^{(m)}$  are also expressed by (5), with the parameters  $(\theta_1, \dots, \theta_{n+2})$  being certain rational multiples of  $\pi$ , distributed in a regular lattice. The mapping from the 2-torus to the eigenlocus is illustrated in Fig. 1.

Another interesting aspect of Example 8 is that all the points  $\mathbf{x} = (x_0, \dots, x_n)$  in the sets  $\mathcal{E}_{\mathcal{A}}^{(m)}$  satisfy the conditions  $x_j = \overline{x_{n-j}}$ ,  $j = 0, 1, \dots, n$ , which explains why we can draw  $C_{\mathcal{A}} \subset \mathbb{C}^2$  in Fig. 1a as a 2-dimensional set. For larger  $n$ ,  $C_{\mathcal{A}}$  is a  $(n+1)$ -dimensional analogue of the two-dimensional deltoid, shown in Fig. 1a.

For general  $\mathcal{A}$ , we do not have the inclusion  $\mathcal{E}_{\mathcal{A}}^{(m)} \subseteq C_{\mathcal{A}}$  for arbitrary finite  $m$ . However, if  $\mathcal{A}$  has an additional extra symmetry, this seems to be case.

**Definition 9.** A banded Toeplitz matrix such that its  $Q(t, \mathbf{x})$  in (3) satisfies

$$\overline{Q(t, x_0, x_1, \dots, x_n)} = \bar{t}^{h+k-1} Q(1/\bar{t}, \bar{x}_n, \bar{x}_{n-1}, \dots, \bar{x}_0)$$

is called multihermitian of order  $n$ .

**Conjecture 10.** If  $\mathcal{A}$  is multihermitian of order  $n$ , then each point  $\mathbf{x} = (x_0, x_1, \dots, x_n) \in \mathcal{E}_{\mathcal{A}}^{(m)}$  satisfies  $x_j = \overline{x_{n-j}}$  for  $j = 0, 1, \dots, n$ .

Conjecture 10 obviously holds for the case  $n = 0$ , as it reduces to the fact that hermitian matrices have real eigenvalues. It is also straightforward to check that Conjecture 10 is true for the Chebyshev case of Example 8 above.

We have extensive numerical evidence for this conjecture. Another strong indication supporting Conjecture 10 is that if  $\mathcal{A}$  is multi-hermitian, then every point  $\mathbf{x} \in C_{\mathcal{A}}$  (which by Conjecture 7 is in the limit eigenlocus) satisfies the required symmetry  $x_j = \overline{x_{n-j}}$  for  $j = 0, 1, \dots, n$ .

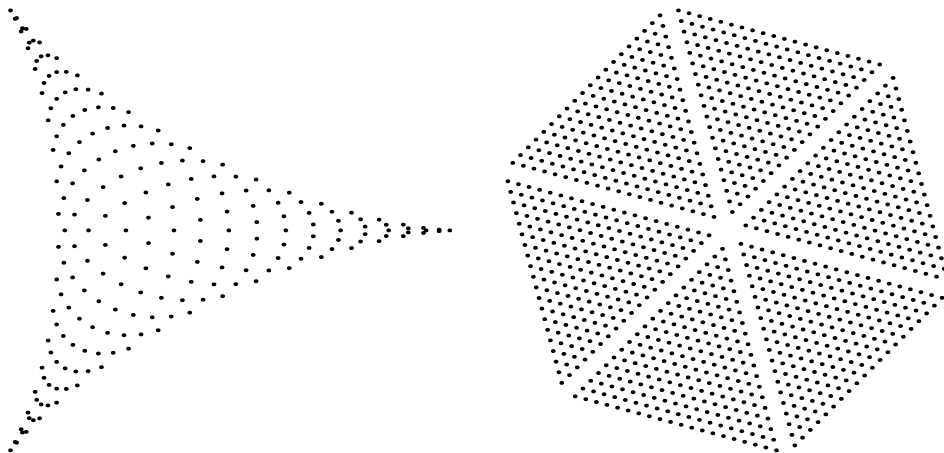
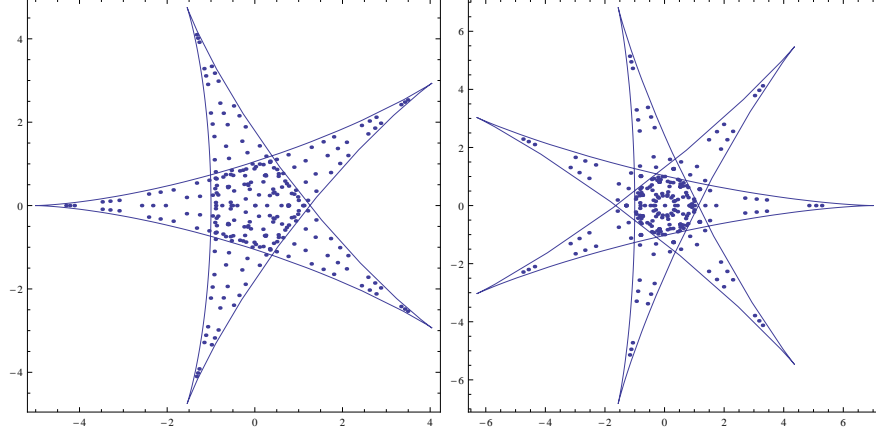


FIGURE 1. The eigenvalue locus  $\mathcal{E}_2^{(20)}$  and its pull-back to  $T^2$ . The torus  $T^2$  is covered with a hexagon, where each triangle is mapped to the eigenlocus. The 6-fold symmetry is due to the  $S_3$ -action by permutation of the arguments  $\theta_1, \theta_2, \theta_3$  in (5). (Notice  $\theta_1 + \theta_2 + \theta_3 = 0$  and this is the subspace which is illustrated in the figure to the right.)

The next group of examples are bivariate analogues of special univariate cases originally studied in [8], and later in [4], where they are referred to as “star-shaped curves”:

**Example 11.** The bivariate case when  $Q(t, \mathbf{x}) = 1 + t^d x_0 + t^{d+1} x_1 + t^{2d+1}$ ,  $d \geq 1$  gives sets in  $\mathbb{C}^2$  where  $x_0 = \overline{x_1}$ , by Conjecture 10. They correspond to Toeplitz


 FIGURE 2. 5-edged star, when  $d = 2$  and 7-edged star, when  $d = 3$ 

matrices of the form

$$\begin{pmatrix} x_0 & x_1 & 1 & 0 & 0 & \cdots \\ 1 & x_0 & x_1 & 1 & 0 & \cdots \\ 0 & 1 & x_0 & x_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \begin{pmatrix} x_0 & x_1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & x_0 & x_1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & x_0 & x_1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \dots$$

The above two matrices represent  $d = 1$  and  $d = 2$ .

Figures 2 and 3 present the distributions of  $x_0 \in \mathbb{C}$ , for  $d = 2, 3, 4$ . (Recall that  $x_1 = \bar{x}_0$ .) The points shown on these figures belong to  $\mathcal{E}_{\mathcal{A}}^{(m)}$  for  $m = 13, 14, 15$ , and the curves are certain hypocycloids, parametrizing the boundary of  $C_{\mathcal{A}}$ . More explicitly, for a given integer  $d \geq 1$  the hypocycloid boundary for  $x_0 \in \mathbb{C}$  is given by

$$x_0 = (-1)^d e^{-i(d+2)\theta} \left( (d+2)e^{i(2d+3)\theta} + d+1 \right) \text{ where } \theta \in [0, 2\pi],$$

which is one of the implications of Conjecture 7.

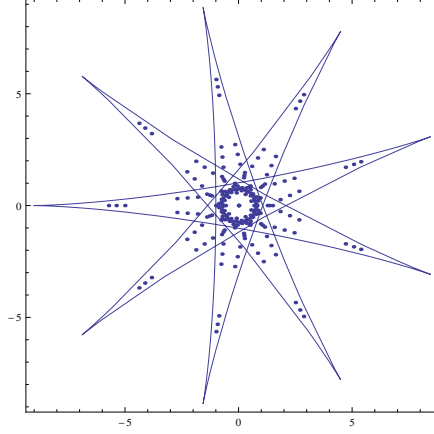
Finally, the main result of this note is as follows;

**Theorem 12.** *For any banded Toeplitz matrix  $\mathcal{A}$ , where  $B_{\mathcal{A}}$  is defined by (2) and  $C_{\mathcal{A}}$  is defined by (4), one has  $B_{\mathcal{A}} \subseteq C_{\mathcal{A}}$ .*

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## 2. PROOFS

*Proof of Proposition 5.* We shall prove items (i) and (ii) simultaneously by calculating the leading homogeneous part of  $P_{\mathcal{A}}^I(x_0, \dots, x_n)$ . Let us order the set of all admissible indices  $I = (1 \leq i_1 < \dots < i_m \leq m+n)$  lexicographically. We can

FIGURE 3. 9-edged star, when  $d = 4$ .

also order lexicographically all monomials of degree  $m$  in  $x_0, \dots, x_n$ . By equation (1)  $P_{\mathcal{A}}^I(x_0, \dots, x_n) = \det A_I$  where the columns of  $A_I$  are indexed by  $I$ . Let  $\tilde{P}_{\mathcal{A}}^I(x_0, \dots, x_n)$  be the homogeneous part of  $P_{\mathcal{A}}^I(x_0, \dots, x_n)$  of degree  $m$ . One can easily see that the product of all entries on the main diagonal of  $A_I$  contains the monomial  $\mathbf{m}_I$  of degree  $m$  given by  $\mathbf{m}_I = \prod_{j=1}^m x_{i_j - j + 1}$ . Moreover it is straightforward that  $\tilde{P}_{\mathcal{A}}^I(x_0, \dots, x_n) = \mathbf{m}_I + \dots$  where  $\dots$  stands for the linear combination of monomials  $\mathbf{m}_{I'}$  of degree  $m$  coming other  $I'$  which are lexicographically smaller than  $I$ . In other words, the matrix formed by  $\tilde{P}_{\mathcal{A}}^I(x_0, \dots, x_n)$  versus monomials is triangular in the lexicographic ordering with unitary main diagonal which proves items (i) and (ii).

Item (iii) is just a reformulation of Proposition 2 above.  $\square$

Throughout the rest of the paper, we use the convention  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{h+k})$ . We will also assume that  $c_h = 1$ , which corresponds to a rescaling of the original matrix  $\mathcal{A}$ . This is equivalent to the assumption that  $Q(t, \mathbf{x})$  is monic. By shifting the variables in  $\mathbf{x}$ , we may also assume, without loss of generality, that  $c_0 = c_1 = \dots = c_n = 0$  in  $\mathcal{A}$ .

In the above notation, it is convenient to work with the roots of  $Q(t, \mathbf{x})$ . This motivates the following definitions. Let  $\Gamma_j \subset \mathbb{C}^{h+k}$ ,  $j = k, \dots, k+n$  denote the real semi-algebraic hypersurface consisting of all  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{h+k})$  such that when the  $\alpha_j$  are ordered with increasing modulus,  $|\alpha_j| = |\alpha_{j+1}|$ . Similarly, let  $G_j$  be defined as the real semi-algebraic set

$$\{\mathbf{x} \in \mathbb{C}^{n+1} : Q(t, \mathbf{x}) = (t - \alpha_1) \dots (t - \alpha_{h+k}) \text{ where } \boldsymbol{\alpha} \in \Gamma_j\}.$$

Then, by definition,  $C_{\mathcal{A}} = \bigcap_{j=k}^{k+n} G_j$ .

**Proposition 13.** *For any banded Toeplitz matrix  $\mathcal{A}$  and any non-negative  $n < h$ , the set  $C_{\mathcal{A}}$  defined by (3)-(4) is compact.*

*Proof.* As discussed above, we may without loss of generality assume that  $c_h = 1$  and  $c_0 = c_1 = \dots = c_n = 0$ . Since  $Q$  may be assumed to be monic, we have  $c_j = e_{h-j}(-\boldsymbol{\alpha})$  for  $-k \leq j < 0$  and  $n < j \leq h$ , and  $x_j = -e_{h-j}(-\boldsymbol{\alpha})$  when

$0 \leq j \leq n$ . Thus, it suffices to show that the set of  $\alpha \in \mathbb{C}^{h+k}$  that satisfies the conditions (3)-(4), is compact. It is also evident that the set  $C_A$  is closed, so we only need to show that it is bounded. We show this fact by contradiction.

Assume we have a sequence of roots  $\{\alpha^m\}_{m=1}^\infty$  of (3) such that  $\|\alpha^m\| \rightarrow \infty$  where (4) is satisfied for each  $\alpha^m$ . We assume that the modulus of the roots are always ordered increasingly. There are two cases to consider.

**Case 1:** Assume that for some  $0 \leq b < k$ , a sequence of individual roots satisfies the condition  $|\alpha_{b+1}^m| \rightarrow \infty$  but  $|\alpha_j^m|$  are bounded for all  $m$  and  $j \leq b$ . Then consider  $e_{h+k-b}(\alpha)$ . Since  $b < k$ , in our notation  $e_{h+k-b}(\alpha)$  equals the coefficient  $c_{b-k}$ . Notice that  $e_{h+k-b}$  contains the term  $\alpha_{b+1}\alpha_{b+2} \cdots \alpha_{h+k}$  which grows quicker than all other terms in the expansion of  $e_{h+k-b}(\alpha)$ . This contradicts the assumption  $e_{h+k-b}(\alpha) = c_{b-k}$ .

**Case 2:** Assume that for some  $b$  with  $k+n \leq b < h+k$ , we have a sequence of individual roots  $|\alpha_{b+1}^m| \rightarrow \infty$  but  $|\alpha_j^m|$  are bounded for all  $m$  and  $j \leq b$ . Consider

$$e_b(\alpha) = e_b(\alpha_1, \dots, \alpha_{h+k}) = \sum_{\sigma \in \binom{[h+k]}{b}} \frac{e_0}{\alpha_{\sigma_1} \alpha_{\sigma_2} \cdots \alpha_{\sigma_b}}.$$

This contains an expression with the denominator  $\alpha_1 \alpha_2 \cdots \alpha_b$ , i.e. the product of all bounded roots. Now, since  $h+k-b$  roots among all  $h+k$  roots grow in absolute value, and the product  $\alpha_1 \cdots \alpha_{h+k}$  equals  $c_h$ , it follows that  $|\alpha_1 \alpha_2 \cdots \alpha_b| \rightarrow 0$  as  $m \rightarrow \infty$ , and this term converges to 0 quicker than any other product  $\alpha_{\sigma_1} \alpha_{\sigma_2} \cdots \alpha_{\sigma_b}$ . Thus,  $|e_b|$  should grow. This contradicts the assumption  $e_b(\alpha) = c_{h-b}$ .

Notice that under our assumptions, the above cases cover all possible ways for a sequence of roots to diverge. Since both cases yield a contradiction, it follows that any sequence of roots of (3) satisfying (4) must be bounded. Thus,  $C_A$  is compact.  $\square$

The following result is multivariate analog of a known fact in the case  $n = 0$ , see [3, Prop. 11.18 and 11.19].

**Proposition 14.** *In the notation of (3)-(4), for any  $\mathbf{x}$  belonging to the boundary  $\partial C_A$  of  $C_A$ , at least one of the following three conditions is satisfied:*

- (i) *the discriminant of  $Q(t, \mathbf{x})$  with respect to  $t$  vanishes, i.e.  $Q(t, \mathbf{x})$  has (at least) a double root in  $t$ .*
- (ii)  $|\alpha_{k-1}(\mathbf{x})| = |\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})|$ .
- (iii)  $|\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})| = |\alpha_{k+n+2}(\mathbf{x})|$ .

*Proof.* We need the following two simple statements.

**Lemma 15.** *Let  $Pol_d$  be the set of all monic polynomials of degree  $d$  with complex coefficients. Let  $\Sigma_{p,q} \subset Pol_d$  be the subset of polynomials satisfying*

$$(8) \quad |\alpha_p| = |\alpha_{p+1}| = \cdots = |\alpha_q|,$$

*where  $1 \leq p < q \leq d$  and  $\alpha_1, \alpha_2, \dots, \alpha_d$  being the roots of polynomials ordered according to their increasing absolute values. Then  $\Sigma_{p,q}$  is a real semi-algebraic set of codimension  $q - p$  whose boundary is the union of three pieces:  $\Sigma_{p-1,q}$ ,  $\Sigma_{p,q+1}$  and the intersection of  $\Sigma_{p,q}$  with the standard discriminant in  $Pol_d$ , i.e. the set of polynomials having multiple roots. (Notice that if  $p = 1$  then  $\Sigma_{p-1,q}$  is empty, and if  $q = d$  then  $\Sigma_{p,q+1}$  is empty by definition.)*

*Proof.*  $\Sigma_{p,q}$  is obtained as the image under the Vieta map of an obvious semi-algebraic set  $|\alpha_1| \leq |\alpha_2| \leq \dots \leq |\alpha_p| = |\alpha_{p+1}| = \dots = |\alpha_q| \leq |\alpha_{q+1}| \leq \dots \leq |\alpha_d|$ . Notice that the Vieta map is a local diffeomorphism outside the preimage of the standard discriminant. Therefore the boundary of  $\Sigma_{p,q}$  must either belong to the standard discriminant or to one of  $\Sigma_{p-1,q}$  or  $\Sigma_{p,q+1}$ . The former is the common boundary between  $\Sigma_{p,q}$  and  $\Sigma_{p-1,q-1}$  and the latter is the common boundary between  $\Sigma_{p,q}$  and  $\Sigma_{p+1,q+1}$ .  $\square$

Given a closed Whitney stratified set  $X$  (for example, semi-analytic) we say that  $X$  is a *k-dimensional stratified set without boundary* if

- (i) the top-dimensional strata of  $X$  have dimension  $k$ ;
- (ii) for any point  $x$  lying in any stratum of dimension  $k - 1$ , one can choose orientation of the (germs of)  $k$ -dimensional strata of a sufficiently small neighborhood of  $x$  in  $X$  so that  $\partial X = 0$ .

**Lemma 16.** *The boundary of the intersection of any closed semi-algebraic set  $\Gamma$  with any closed algebraic set  $\Theta$  is included in the intersection of the boundary  $\partial\Gamma$  with  $\Theta$ .*

*Proof.* Observe that any real algebraic variety  $X$  of dimension  $k$  is a stratifiable set without boundary. Indeed, the fact we are proving is local, and it suffices to prove it for generic  $x$  on  $(k - 1)$ -dimensional strata.

Consider an embedding of  $X$  in some high-dimensional linear space, take the Whitney stratification with  $x$  on its stratum  $Y \subset B$  of dimension  $k - 1$ , and a transversal to  $Y$  of codimension  $k - 1$  at  $x$ .

Therefore, we may now assume that the germ of  $X$  near  $x$  is topologically a product of a germ of algebraic curve and a germ of a smooth manifold of dimension  $k - 1$ . Furthermore, a germ of any real algebraic curve  $\Gamma$  can be always oriented so that  $\partial\Gamma = 0$  which follows from the existence of Puiseux series for an arbitrary branch of algebraic curve. This argument shows that any point in the intersection  $\Gamma \cap \Theta$  which does not belong to the boundary of  $\Gamma$  can not lie on the boundary of this intersection which settles Lemma 16.  $\square$

Lemmas 15 and 16 immediately imply Proposition 14 since every  $C_A$  is the intersection of an appropriate  $\Sigma_{p,q}$  with an appropriate affine subspace in  $Pol_{k+h}$ .  $\square$

*Proof of Theorem 12.* In our notation, let  $D_j^m(\mathbf{x})$  be the determinant of the  $m \times m$ -matrix  $A_I$  with  $I = \{j + 1, j + 2, \dots, j + m\}$  for  $0 \leq j \leq n$ . It is evident that  $\mathcal{E}_A^{(m)}$  is a subset of the set  $\tilde{\mathcal{E}}_A^{(m)}$  of solutions to the system of polynomial equations

$$(9) \quad D_0^m(\mathbf{x}) = D_1^m(\mathbf{x}) = \dots = D_n^m(\mathbf{x}) = 0.$$

We will show a stronger statement that, in notation of Theorem 12,

$$\lim_{m \rightarrow \infty} \tilde{\mathcal{E}}_A^{(m)} \subseteq C_A.$$

Although each individual  $\tilde{\mathcal{E}}_A^{(m)}$  (considered as a points set with multiplicities) is strictly bigger than  $\mathcal{E}_A^{(m)}$  the limits  $B_A = \lim_{m \rightarrow \infty} \mathcal{E}_A^{(m)}$  and  $\lim_{m \rightarrow \infty} \tilde{\mathcal{E}}_A^{(m)}$  seem to coincide as infinite sets.

The next proposition accomplishes the proof of Theorem 12.  $\square$



In Theorem 4 of [1] it was shown that each sequence of determinants  $\{D_j^m(\mathbf{x})\}_{m=1}^\infty$  as above satisfies a linear recurrence relation with coefficients depending on  $\mathbf{x}$ . The characteristic polynomial  $\chi_j(t)$  of the  $j$ -th recurrence can be factorized as

(10)

$$\chi_j(t, \mathbf{x}) = \prod_{\sigma} (t - r_{j\sigma}), \text{ where } r_{j\sigma} = (-1)^{k+j} (\alpha_{\sigma_1} \cdots \alpha_{\sigma_{k+j}})^{-1}, \sigma \in \binom{[k+h]}{k+j}.$$

**Proposition 17.** *Suppose that  $\{\mathbf{x}_m\}_1^\infty$  is a sequence of points in  $\mathbb{C}^{n+1}$  satisfying the system of equations:*

$$(11) \quad D_j^m(\mathbf{x}_m) = 0 \text{ for } j = 0, 1, \dots, n \text{ and } m = 1, 2, \dots$$

*and such that the limit  $\lim_{m \rightarrow \infty} \mathbf{x}_m =: \mathbf{x}^*$  exists. Then for all  $j = 0, \dots, n$   $|\alpha_{k+j}(\mathbf{x}^*)| = |\alpha_{k+j+1}(\mathbf{x}^*)|$  when the  $\alpha_i$  are indexed with increasing order of their modulus.*

*Proof.* Provided that all the roots of  $\chi_j(t, \mathbf{x})$  are distinct, by using a version of Widom's formula, (see [1, 3]) we have

$$(12) \quad D_j^m(\mathbf{x}) = \sum_{\sigma} \prod_{l \in \sigma, i \notin \sigma} \left(1 - \frac{\alpha_l(\mathbf{x})}{\alpha_i(\mathbf{x})}\right)^{-1} \cdot r_{j\sigma}(\mathbf{x})^m.$$

We may assume that for  $\mathbf{x}^*$  and fixed  $j$ , the  $r_{j\sigma}(\mathbf{x}^*)$  are ordered decreasingly with respect to their modulus (for some ordering  $\sigma = 1, 2, \dots$ ). The goal is to prove that  $|r_{j1}(\mathbf{x}^*)| = |r_{j2}(\mathbf{x}^*)|$  since this implies  $|\alpha_{k+j}(\mathbf{x}^*)| = |\alpha_{k+j+1}(\mathbf{x}^*)|$ . We show this fact by contradiction.

Assume that  $|r_{j1}(\mathbf{x}^*)| > |r_{j2}(\mathbf{x}^*)| \geq \dots \geq |r_{jb}(\mathbf{x}^*)|$ , i.e. that the largest root is simple and has modulus strictly larger than any other root of the characteristic equation (10). By examining (12), it is evident that  $r_{j1}(\mathbf{x}_m)^m$  is the dominating term for sufficiently large  $m$ , that is,  $D_j^m(\mathbf{x}_m)/r_{j1}(\mathbf{x}_m)^m \rightarrow L \neq 0$  as  $m \rightarrow \infty$ .

By standard properties of linear recurrences, this holds even when there are multiple zeros among the smaller roots; remember that our assumption was that  $r_{j1}(\mathbf{x}_m)$  is a simple zero of (10) when  $m$  is large enough.

Hence, for sufficiently large  $m$ ,  $D_j^m(\mathbf{x}_m) \approx L r_{j1}(\mathbf{x}_m)^m$ , which is non-zero for sufficiently large  $m$ . This contradicts the condition that  $\mathbf{x}_m$  satisfies (11). Consequently,  $|r_{j1}(\mathbf{x}^*)| = |r_{j2}(\mathbf{x}^*)|$  for  $j = 0, 1, \dots, n$  and this implies Proposition 17.  $\square$

Proposition 17 implies that  $\mathbf{x}$  lies in  $B_{\mathcal{A}}$  only if  $\mathbf{x}$  is a limit of solutions to (11), but such limit  $\mathbf{x}$  must satisfy that  $|\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \dots = |\alpha_{k+n+1}(\mathbf{x})|$ . Therefore,  $B_{\mathcal{A}} \subseteq C_{\mathcal{A}}$ .

### 3. FURTHER DIRECTIONS

1. It seems relatively easy to describe the stratified structure of  $C_{\mathcal{A}}$  at least in case of generic  $\mathcal{A}$ . In particular, in the Chebyshev case of Example 8 the set  $C_{\mathcal{A}}$  has the same stratification as a simplex of corresponding dimension. One can also understand the stratified structure of the sets  $\Sigma_{p,q}$  introduced in Lemma 15. Since each  $C_{\mathcal{A}}$  is obtained from a corresponding  $\Sigma_{p,q}$  by intersecting it with an affine subspace the stratified structure of the former for generic  $\mathcal{A}$  is also describable. On the other hand, our Example 11 seems to show more complicated stratified structure due to the presence of additional symmetry.

**2.** We say that an (infinite) complex-valued matrix  $\mathcal{A}$  has a *weak univariate orthogonality property* if the sequence of characteristic polynomials of its principal minors obeys the standard 3-term recurrence relation with complex coefficients. There is a straightforward version of this notion for finite square matrices. Obviously, any Jacobi matrix has this property. However, it seems that for any  $m \geq 3$  the set  $WO_m \subset Mat(m, m)$  of all  $m \times m$ -matrices with the latter property has a bigger dimension than the set  $Jac_m \subset Mat(m, m)$  of all Jacobi  $m \times m$ -matrices.

**Problem 18.** *Find the dimension of  $WO_m$ ?*

**3.** Analogously, given a non-negative integer  $n$ , we say that an (infinite) complex-valued matrix  $\mathcal{A}$  has a *weak  $n$ -variate orthogonality property* if the above family  $\{P_{\mathcal{A}}^I(x_0, x_1, \dots, x_n)\}$  (see Definition 4) satisfies the 3-term recurrence relation (2.2) of Theorem 2.1 of [13] with complex coefficients.

There are many similarities between families  $\{P_{\mathcal{A}}^I(x_0, x_1, \dots, x_n)\}$  and families of multivariate orthogonal polynomials which by one of the standard definitions of such polynomials also satisfy (2.2) of Theorem 2.1 of [13] with real coefficients.

Our computer experiments show that in this aspect the case  $n > 0$  is quite different from the classical case  $n = 0$ . In particular, we believe that the following conjecture holds.

**Conjecture 19.** *Given  $n > 0$ , a banded matrix  $\mathcal{A}$  has a weak  $n$ -variate orthogonality property if it is of the form*

$$\mathcal{A} = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n+1} & 0 & 0 & 0 & \dots \\ d_{-1} & d_0 & d_1 & \dots & d_n & d_{n+1} & 0 & 0 & \dots \\ 0 & d_{-1} & d_0 & \dots & d_{n-1} & d_n & d_{n+1} & 0 & \dots \\ 0 & 0 & d_{-1} & \dots & d_{n-2} & d_{n-1} & d_n & d_{n+1} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where  $a_0, \dots, a_{n+1}, d_{-1}, \dots, d_{n+1} \in \mathbb{C}$ .

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DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, S-10691, STOCKHOLM, SWEDEN  
*E-mail address:* `per@math.su.se`

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, S-10691, STOCKHOLM, SWEDEN  
*E-mail address:* `shapiro@math.su.se`